§1.6: THE SECOND DERIVATIVE

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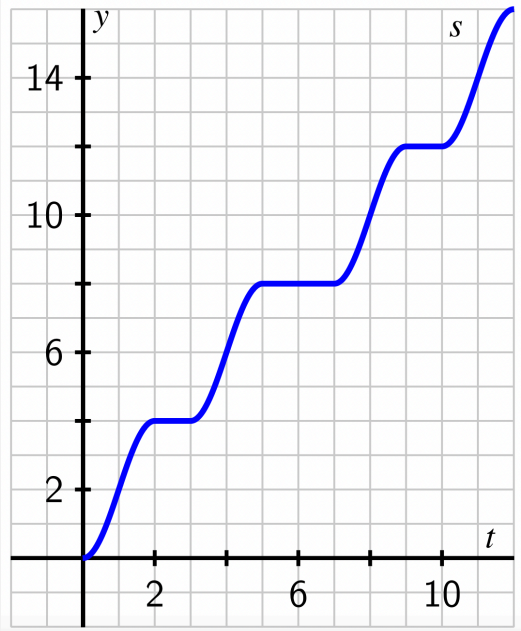
Lecture 6

What information does the derivative
of the derivative give us?


y
s
14
10
6
2
t
2
6
10

Definition

Given a function f defined on the interval (a, b), we say that f is increasing on (a, b) provided that for all x, y satisfying $a < x < y < b$ that $f(x) < f(y)$. Similarly, f is decreasing if for all x, y satisfying $a < x < y < b$ we have $f(y) < f(x)$.

Definition

The second derivative of a function $y = f(x)$ is defined by

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}.$$

Desmos Example

Definition

Let f be differentiable on the interval (a, b). We say f is concave up if and only if $f'(x)$ is increasing on (a, b) (and thus, $f''(x) > 0$ on (a, b)). We say f is concave down if and only if $f'(x)$ is decreasing on (a, b) (i.e., $f''(x) < 0$ on (a, b)).

ACTIVITY 1.6.2

The position of a car driving along a straight road at time t in minutes is given by the function $y = s(t)$ that is pictured in Figure 1. The car's position function has units measured in thousands of feet.

(a) On what intervals is the position function $y = s(t)$ increasing? decreasing? Why?

(b) On which intervals is the velocity function $y = v(t) = s'(t)$ increasing? decreasing? neither? Why?

(c) *Acceleration* is defined to be the instantaneous rate of change of velocity, as the acceleration of an object measures the rate at which the velocity of the object is changing. Say that the car's acceleration function is named $a(t)$. How is $a(t)$ computed from $v(t)$? How is $a(t)$ computed from $s(t)$? Explain.

(d) What can you say about s'' whenever s' is increasing? Why?

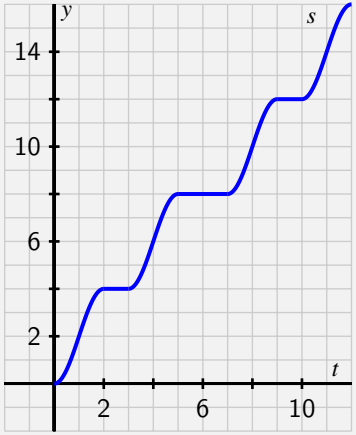


Figure 1: The graph of $y = s(t)$.

(e) Using only the words *increasing*, *decreasing*, *constant*, *concave up*, *concave down*, and *linear*, complete the following sentences. For the position function s with velocity v and acceleration a,

- on an interval where v is positive, s is increasing.
- on an interval where v is negative, s is decreasing.
- on an interval where v is zero, s is constant.
- on an interval where a is positive, v is increasing.
- on an interval where a is negative, v is decreasing.
- on an interval where a is zero, v is constant.
- on an interval where a is positive, s is concave up.
- on an interval where a is negative, s is concave down.
- on an interval where a is zero, s is linear.

ACTIVITY 1.6.3

A potato is placed in an oven, and the potato's temperature F (in degrees Fahrenheit) at various points in time is taken and recorded in the following table. Time t is measured in minutes. In Activity 1.5.2, we computed approximations to $F'(30)$ and $F'(60)$ using central differences. Those values and more are provided in the second table below, along with several others computed in the same way.

t	$F(t)$
0	70
15	180.5
30	251
45	296
60	324.5
75	342.8
90	354.5

t	$F'(t)$
0	NA
15	6.03
30	3.85
45	2.45
60	1.56
75	1.00
90	NA

(a) What are the units on the values of $F'(t)$?

(b) Use a central difference to estimate the value of $F''(30)$.

(c) What is the meaning of the value of $F''(30)$ that you have computed in (b) in terms of the potato's temperature? Write several careful sentences that discuss, with appropriate units, the values of $F(30)$, $F'(30)$, and $F''(30)$, and explain the overall behavior of the potato's temperature at this point in time.

(d) Overall, is the potato's temperature increasing at an increasing rate, increasing at a constant rate, or increasing at a decreasing rate? Why?

This activity builds on our experience and understanding of how to sketch the graph of f' given the graph of f. In Figure 2, given the respective graphs of two different functions f, sketch the corresponding graph of f' on the first axes below, and then sketch f'' on the second set of axes. In addition, for each, write several careful sentences in the spirit of those in Activity 1.6.2 that connect the behaviors of f, f', and f''. For instance, write something such as

f' is _____ on the interval _____, which is connected to the fact that f is _____ on the same interval _____, and f'' is _____ on the interval as well.

but of course with the blanks filled in. Throughout, view the scale of the grid for the graph of f as being 1×1, and assume the horizontal scale of the grid for the graph of f' is identical to that for f. If you need to adjust the vertical scale on the axes for the graph of f' or f'', you should label that accordingly.

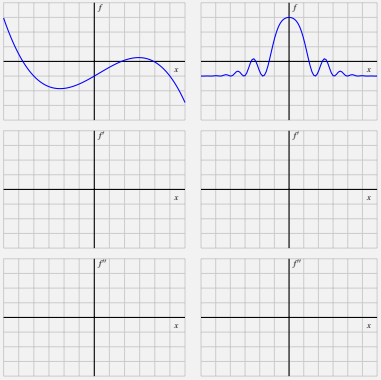


Figure 2: Two given functions f, with axes provided for plotting f' and f'' below.