§1.7: LIMITS, CONTINUITY, AND DIFFERENTIABILITY

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Lecture 7

What does it mean graphically for $f'(a)$ to exist, and how is this related to the continuity of f at $x = a$?

Preview Activity (Desmos)

Definition

We say that f has limit L_1 as x approaches a from the left if we can make the value of $f(x)$ as close to L_1 as we like by taking x sufficiently close (but not equal) to a while having $x < a$. We write

$$\lim_{x \to a^-} f(x) = L_1.$$

We say that f has limit L_2 as x approaches a from the right if we can make the value of $f(x)$ as close to L_2 as we like by taking x sufficiently close (but not equal) to a while having $x > a$. We write

$$\lim_{x \to a^+} f(x) = L_2.$$

Then $\lim_{x \to a} f(x) = L$ if and only if $L_1 = L_2 = L$.

Desmos Example 1.7.1

ACTIVITY 1.7.2

Consider a function that is piecewise-defined according to the formula

$$f(x) = \begin{cases} 3(x+2)+2 & \text{for } -3 < x < -2 \\ \frac{2}{3}(x+2)+1 & \text{for } -2 \leq x < -1 \\ \frac{2}{3}(x+2)+1 & \text{for } -1 < x < 1 \\ 2 & \text{for } x = 1 \\ 4 - x & \text{for } x > 1 \end{cases}$$

Use the given formula to answer the following questions.

(a) For each of the values $a = -2, -1, 0, 1, 2$, compute $f(a)$.

(b) For each of the values $a = -2, -1, 0, 1, 2$, determine $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$.

(c) For each of the values $a = -2, -1, 0, 1, 2$, determine $\lim_{x \to a} f(x)$. If the limit fails to exist, explain why by discussing the left- and right-hand limits at the relevant a-value.

(d) For which values of a is the following statement true?

$$\lim_{x \to a} f(x) \neq f(a)$$

(e) On the axes provided, sketch an accurate, labeled graph of $y = f(x)$. Be sure to carefully use open circles ($\circ$) and filled circles ($\bullet$) to represent key points on the graph, as dictated by the piecewise formula.

What does it mean for a function to be continuous at the point $x = a$?

Definition

A function f is continuous at $x = a$ provided that

1. $\lim_{x \to a} f(x)$ *exists*
2. $f(a)$ *exists*
3. $\lim_{x \to a} f(x) = f(a)$

ACTIVITY 1.7.3

This activity builds on your work in Preview Activity 1.7.1, using the same function f as given by the graph that is repeated in Figure 1

(a) At which values of a does $\lim_{x \to a} f(x)$ not exist?

(b) At which values of a is $f(a)$ not defined?

(c) At which values of a does f have a limit, but $\lim_{x \to a} f(x) \neq f(a)$?

(d) State all values of a for which f is not continuous at $x = a$.

(e) Which condition is stronger, and hence implies the other: f has a limit at $x = a$ or f is continuous at $x = a$? Explain, and hence complete the following sentence: "If f ______________________ at $x = a$, then f ______________________ at $x = a$," where you complete the blanks with *has a limit* and *is continuous*, using each phrase once.

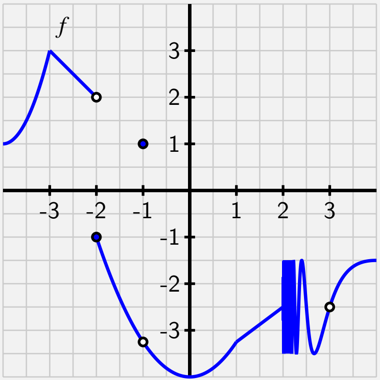

Figure 1: The graph of $y = f(x)$ for Activity 1.7.3.

ACTIVITY 1.7.4

In this activity, we explore two different functions and classify the points at which each is not differentiable. Let g be the function given by the rule $g(x) = |x|$, and let f be the function that we have previously explored in Preview Activity 1.7.1, whose graph is given again in Figure 2.

(a) Reasoning visually, explain why g is differentiable at every point x such that $x \neq 0$.

(b) Use the limit definition of the derivative to show that $g'(0) = \lim_{h \to 0} \frac{|h|}{h}$.

(c) Explain why $g'(0)$ fails to exist by using small positive and negative values of h.

(d) State all values of a for which f is not differentiable at $x = a$. For each, provide a reason for your conclusion.

(e) True or false: if a function p is differentiable at $x = b$, then $\lim_{x \to b} p(x)$ must exist. Why?

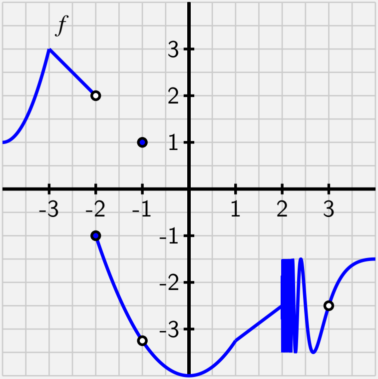

Figure 2: The graph of $y = f(x)$ for Activity 1.7.4.