# §2.6: DERIVATIVES OF INVERSE FUNCTIONS

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Lecture 14

How can we use the function-inverse relationship to calculate derivatives of inverses?

# Preview Activity Discussion

## ON INVERSES

Question: What must be true of $y = f(x)$ for $f$ to have an inverse $f^{-1}$?

We have $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

# Derivative of natural log

# ACTIVITY 2.6.2

(a) $h(x) = x^2 \ln(x) \Rightarrow h'(x) = x^2 \cdot \frac{1}{x} + 2x \cdot \ln(x)$

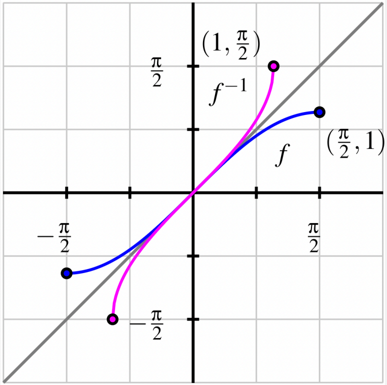
(b) $p(t) = \frac{\ln(t)}{e^t+1} \Rightarrow p'(t) = \frac{(e^t+1)\frac{1}{t} - \ln(t)e^t}{(e^t+1)^2}$

(c) $s(y) = \ln(\cos(y) + 2) \Rightarrow s'(y) = \frac{1}{\cos(y)+2} \cdot (-\sin(y))$

(d) $z(x) = \tan(\ln(x)) \Rightarrow z'(x) = \sec^2(\ln(x)) \cdot \frac{1}{x}$

(e) $m(z) = \ln(\ln(z)) \Rightarrow m'(z) = \frac{1}{\ln(z)} \cdot \frac{1}{z}$

# INVERSE TRIG FUNCTIONS




Thus, for $-1 < x < 1$:

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\sqrt{1 - x^2}}.$$

# ACTIVITY 2.6.3

The following prompts in this activity will lead you to develop the derivative of the inverse tangent function.

(a) Let $r(x) = \arctan(x)$. Use the relationship between the arctangent and tangent functions to rewrite this equation using only the tangent function.

(b) Differentiate both sides of the equation you found in (a). Solve the resulting equation for $r'(x)$, writing $r'(x)$ as simply as possible in terms of a trigonometric function evaluated at $r(x)$.

(c) Recall that $r(x) = \arctan(x)$. Update your expression for $r'(x)$ so that it only involves trigonometric functions and the independent variable $x$.

(d) Introduce a right triangle with angle $\theta$ so that $\theta = \arctan(x)$. What are the three sides of the triangle?

(e) In terms of only $x$ and 1, what is the value of $\cos(\arctan(x))$?

(f) Use the results of your work above to find an expression involving only 1 and $x$ for $r'(x)$.

# ACTIVITY 2.6.4

(a) $f(x) = x^3 \arctan(x) + e^x \ln(x) \Rightarrow f'(x) = x^3 \cdot \frac{1}{1+x^2} + 3x^2 \arctan(x) + e^x \cdot \frac{1}{x} + e^x \ln(x)$

(b) $p(t) = 2^{t \arcsin(t)} \Rightarrow p'(t) = \ln(2) 2^{t \arcsin(t)} \cdot \left( \arcsin(t) + t \cdot \frac{1}{\sqrt{1-t^2}} \right)$

(c) $h(z) = [\arcsin(5z) + \arctan(4-z)]^{27} \Rightarrow h'(z) =$
$27 [\arcsin(5z) + \arctan(4-z)]^{26} \cdot \left( \frac{1}{\sqrt{1-(5z)^2}} \cdot 5 + \frac{1}{1+(4-z)^2} \cdot (-1) \right)$

(d) $s(y) = \cot(\arctan(y)) = \frac{1}{y} \Rightarrow s'(y) = -\frac{1}{y^2}$

(e) $m(v) = \ln(\sin^2(v) + 1) \Rightarrow m'(v) = \frac{1}{\sin^2(v)+1} \cdot (2 \sin(v) \cos(v))$

(f) $g(w) = \arctan\left( \frac{\ln(w)}{1+w^2} \right) \Rightarrow g'(w) = \frac{1}{1 + \left( \frac{\ln(w)}{1+w^2} \right)^2} \cdot \left[ \frac{(1+w^2) \cdot \frac{1}{w} - \ln(w)(2w)}{(1+w^2)^2} \right]$

# IN GENERAL

**Theorem (Inverse Function Theorem)**

*Let $f(x)$ be differentiable, $g$ the inverse of $f$, $f(a) = b$, and $f'(a) \neq 0$. Then:*

$$g'(b) = \frac{1}{f'(a)}.$$