§2.8: L'HÔPITAL'S RULES AND LIMITS AT INFINITY

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Lecture 16

How can we use derivatives to evaluate indeterminate limits?

Preview Activity Discussion (Desmos)

Theorem

Let f and g be differentiable functions at $x = a$, and suppose $f(a) = g(a) = 0$ and $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

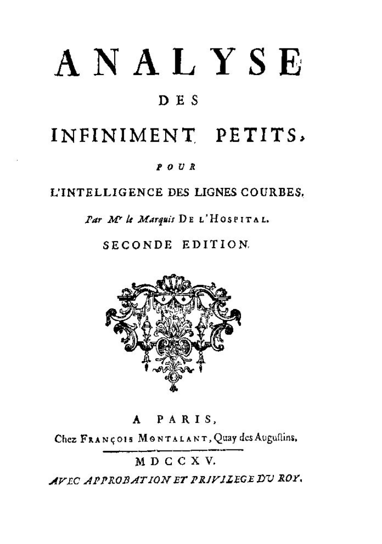
More generally:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

GUILLAUME DE L'HÔPITAL

- 1661-1704
- First calculus textbook: *Infinitesimal calculus with applications to curved lines*
- Stigler's law of eponymy at work

Johann Bernoulli


ANALYSE
DES
INFINIMENT PETITS,
POUR
L'INTELLIGENCE DES LIGNES COURBES.
Par Mr le Marquis DE L'HOSPITAL.
SECONDE EDITION.

A PARIS,
Chez FRANÇOIS MONTALANT, Quay des Auguſtins.
M D C C X V.
AVEC APPROBATION ET PRIVILEGE DU ROY.

ACTIVITY 2.8.2

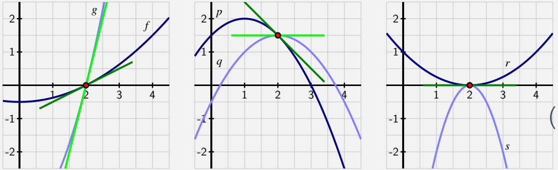
(a) $\lim_{x\to 0} \frac{\ln(1+x)}{x} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1$

(b) $\lim_{x\to \pi} \frac{\cos(x)}{x} = \frac{-1}{\pi}$

(c) $\lim_{x\to 1} \frac{2\ln(x)}{1-e^{x-1}} \overset{\text{L'H}}{=} \lim_{x\to 1} \frac{\frac{2}{x}}{-e^{x-1}} = \frac{2}{-1} = -2$

(d) $\lim_{x\to 0} \frac{\sin(x)-x}{\cos(2x)-1} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\cos(x)-1}{-2\sin(2x)} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{-\sin(x)}{-4\cos(2x)} = 0$

ACTIVITY 2.8.3



(a) Use the left-hand graph to determine the values of $f(2)$, $f'(2)$, $g(2)$, and $g'(2)$. Then, evaluate

$$\lim_{x \to 2} \frac{f(x)}{g(x)} = \frac{1}{8}$$

(b) Use the middle graph to find $p(2)$, $p'(2)$, $q(2)$, and $q'(2)$. Then, determine the value of

$$\lim_{x \to 2} \frac{p(x)}{q(x)} = 1$$

(c) Use the right-hand graph to compute $r(2)$, $r'(2)$, $s(2)$, $s'(2)$. Explain why you cannot determine the exact value of

$$\lim_{x \to 2} \frac{r(x)}{s(x)}$$

without further information being provided, but that you can determine the sign of $\lim_{x \to 2} \frac{r(x)}{s(x)}$. In addition, state what the sign of the limit will be, with justification.

Two ways:

- $x \to \pm\infty$
- $f(x) \to \pm\infty$

Desmos Example 2.8.1

Theorem

If f, g are differentiable and both approach 0 or $\pm\infty$ as $x \to a$ (where $-\infty \leq a \leq \infty$), then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Example:

$$\lim_{x \to \infty} \frac{x^2}{e^x}$$

ACTIVITY 2.8.4

(a) $\lim_{x\to\infty} \frac{x}{\ln x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1}{1/x} = \lim_{x\to\infty} x = \infty$

(b) $\lim_{x\to\infty} \frac{e^x+x}{2e^x+x^2} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x+1}{2e^x+2x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x}{2e^x+2} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x}{2e^x} = \frac{1}{2}$

(c) $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0$

(d) $\lim_{x\to \frac{\pi}{2}^-} \frac{\tan(x)}{x-\frac{\pi}{2}} = -\infty$

(e) $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$

Calculation Exam Prep

CALCULATION EXAM 2: FALL 2022

1. $f(x) = 8x^4 + ex^3 + 2x^{\frac{1}{2}} + 1 \Rightarrow f'(x) = 32x^3 + 3ex^2 + \frac{1}{2} \cdot 2x^{-1/2}$

2. $g(y) = 12^5 - \ln(y) \Rightarrow g'(y) = -\frac{1}{y}$

3. $h(t) = \dfrac{\csc(t) + \csc(t)t^2}{1 + t^2} = \csc(t) \Rightarrow h'(t) = -\csc(t)\cot(t)$

4. $l(s) = \sin(s)\cos(s) \Rightarrow l'(s) = \cos(s)\cos(s) - \sin(s)\sin(s)$

5. $k(w) = \frac{2w^3+3w}{3w^2+2w} \Rightarrow k'(w) = \frac{(3w^2+2w)(6w^2+3)-(2w^3+3w)(6w+2)}{(3w^2+2w)^2}$

6. $f(y) = \arctan(\sin(y)) \Rightarrow f'(y) = \frac{1}{1+(\sin(y))^2}\cos(y)$

7. $g(x) = e^{\ln(x)x} \Rightarrow g'(x) = e^{\ln(x)x}\left(\frac{1}{x}x + \ln(x)\right)$

8. $h(t) = \sqrt[4]{t^2 - 12t + \pi} + (\sin^2(t) + 21)^{-7} \Rightarrow h'(t) =$
$\frac{1}{4}\left(t^2 - 12t + \pi\right)^{-3/4}(2t - 12) + (-7)(\sin^2(t) + 21)^{-8}(2\sin(t)\cos(t))$

9. $l(s) = 12^{e^{\tan(s)}} \Rightarrow l'(s) = \ln(12)12^{e^{\tan(s)}}e^{\tan(s)}\sec^2(s)$

10. $k(w) = \frac{(4-w)^2(1+2w)}{(3+9w)} \Rightarrow k'(w) = \frac{(3+9w)[(4-w)^2(2)+(1+2w)\cdot 2(4-w)(-1)]-(4-w)^2(1+2w)(9)}{(3+9w)^2}$