§3.1: RELATED RATES

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Lecture 16

When related quantities change over time, how do their rates of change relate to one another?

PREVIEW ACTIVITY DISCUSSION

STRATEGY

1. Identify the quantities that are changing over time t
2. Find the rates of change of the changing quantities
3. Find an equation that relates the changing quantities
4. Differentiate implicitly with respect to time
5. Evaluate at the relevant instant

ACTIVITY 3.1.2

A water tank has the shape of an inverted circular cone (point down) with a base of radius 6 feet and a depth of 8 feet. Suppose that water is being pumped into the tank at a constant instantaneous rate of 4 cubic feet per minute.

(a) Draw a picture of the conical tank, including a sketch of the water level at a point in time when the tank is not yet full. Introduce variables that measure the radius of the water's surface and the water's depth in the tank, and label them on your figure.

(b) Say that r is the radius and h the depth of the water at a given time, t. What equation relates the radius and height of the water, and why? $r = \frac{3}{4}h$

(c) Determine an equation that relates the volume of water in the tank at time t to the depth h of the water at that time. $V = \frac{3}{16}\pi h^3$

(d) Through differentiation, find an equation that relates the instantaneous rate of change of water volume with respect to time to the instantaneous rate of change of water depth at time t. $\frac{dV}{dt} = \frac{9}{16}\pi h^2 \frac{dh}{dt}$

(e) Find the instantaneous rate at which the water level is rising when the water in the tank is 3 feet deep.
$\left.\frac{dh}{dt}\right|_{h=3} = \frac{64}{81\pi} \approx 0.2515$ ft/min

(f) When is the water rising most rapidly: at $h = 3$, $h = 4$, or $h = 5$?

ACTIVITY 3.1.3

A television camera is positioned 4000 feet from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. In addition, the auto-focus of the camera has to take into account the increasing distance between the camera and the rocket. We assume that the rocket rises vertically.

(a) Draw a figure that summarizes the given situation. What parts of the picture are changing? What parts are constant? Introduce appropriate variables to represent the quantities that are changing.

(b) Find an equation that relates the camera's angle of elevation to the height of the rocket, and then find an equation that relates the instantaneous rate of change of the camera's elevation angle to the instantaneous rate of change of the rocket's height (where all rates of change are with respect to time). $\frac{dh}{dt} = 4000\sec^2(\theta)\frac{d\theta}{dt}$

(c) Find an equation that relates the distance from the camera to the rocket to the rocket's height, as well as an equation that relates the instantaneous rate of change of distance from the camera to the rocket to the instantaneous rate of change of the rocket's height (where all rates of change are with respect to time). $h\frac{dh}{dt} = z\frac{dz}{dt}$

(d) Suppose that the rocket's speed is 600 ft/sec at the instant it has risen 3000 feet. How fast is the distance from the television camera to the rocket changing at that moment? If the camera is following the rocket, how fast is the camera's angle of elevation changing at that same moment? $\left.\frac{d\theta}{dt}\right|_{h=3000} = \frac{12}{125}$

(e) If from an elevation of 3000 feet onward the rocket continues to rise at 600 feet/sec, will the rate of change of distance with respect to time be greater when the elevation is 4000 feet than it was at 3000 feet, or less? Why? 4000, but why?

ACTIVITY 3.1.4

As pictured in the applet at `http://gvsu.edu/s/9q`, a skateboarder who is 6 feet tall rides under a 15 foot tall lamppost at a constant rate of 3 feet per second. We are interested in understanding how fast his shadow is changing at various points in time.

(a) Draw an appropriate right triangle that represents a snapshot in time of the skateboarder, lamppost, and his shadow. Let x denote the horizontal distance from the base of the lamppost to the skateboarder and s represent the length of his shadow. Label these quantities, as well as the skateboarder's height and the lamppost's height on the diagram.

(b) Observe that the skateboarder and the lamppost represent parallel line segments in the diagram, and thus similar triangles are present. Use similar triangles to establish an equation that relates x and s. $3s = 2x$

(c) Use your work in (b) to find an equation that relates $\frac{dx}{dt}$ and $\frac{ds}{dt}$. $3\frac{ds}{dt} = 2\frac{dx}{dt}$

(d) At what rate is the length of the skateboarder's shadow increasing at the instant the skateboarder is 8 feet from the lamppost? $\left.\frac{ds}{dt}\right|_{x=8} = 2$ feet per second

(e) As the skateboarder's distance from the lamppost increases, is his shadow's length increasing at an increasing rate, increasing at a decreasing rate, or increasing at a constant rate? Constant

(f) Which is moving more rapidly: the skateboarder or the tip of his shadow? Explain, and justify your answer. The tip of his shadow.

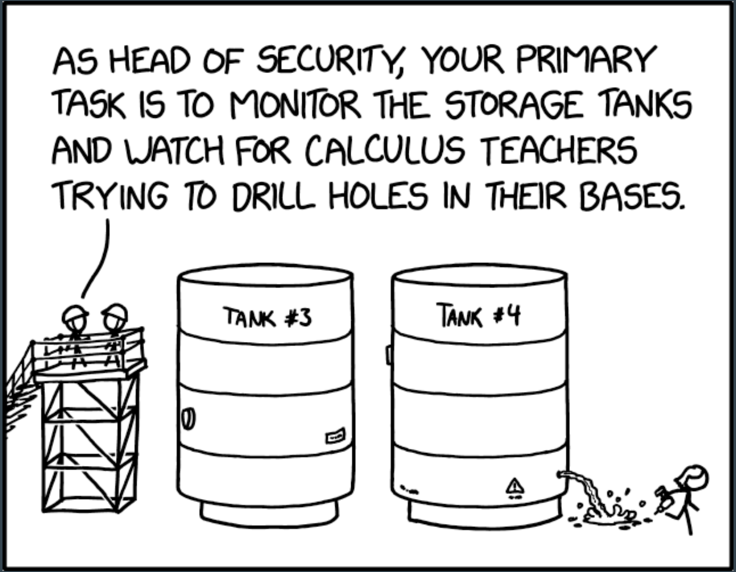
ACTIVITY 3.1.5

A baseball diamond is 90′ square. A batter hits a ball along the third base line and runs to first base. At what rate is the distance between the ball and first base changing when the ball is halfway to third base, if at that instant the ball is traveling 100 feet/sec? At what rate is the distance between the ball and the runner changing at the same instant, if at the same instant the runner is 1/8 of the way to first base running at 30 feet/sec?

- $\left.\dfrac{dz}{dt}\right|_{x=45} = \dfrac{100}{\sqrt{5}} \approx 44.7214$ ft/s
- $\left.\dfrac{ds}{dt}\right|_{x=45} = \dfrac{430}{\sqrt{17}} \approx 104.2903$ ft/s

BONUS

A water tank is in the shape of an inverted cone with depth 10 meters and top radius 8 meters. Water is flowing into the tank at 0.1 cubic meters/min but leaking out at a rate of $0.004h^2$ cubic meters/min, where h is the depth of the water in meters. Can the tank ever overflow?



https://xkcd.com/2974