THEOREMS ABOUT DIFFERENTIABLE FUNCTIONS

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Bonus

Theorem

If f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$, then there exists a number c, with $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Consequence: If $f(a) = f(b) = 0$, then for some c satisfying $a < c < b$, $f'(c) = 0$.

Problem.

If f is differentiable and $f(0) < f(1)$, then there is a number c, with $0 < c < 1$, such that $f'(c) > 0$.

Theorem

Suppose that f is continuous on $a \leq x \leq b$ and differentiable on $a < x < b$. If $f'(x) = 0$ on $a < x < b$, then f is constant on $a \leq x \leq b$.

Proof.

By MVT, given any x_1, x_2 satisfying $a \leq x_1 < x_2 \leq b$, there exists a c satisfying $x_1 < c < x_2$ for which $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$. But $f'(c) = 0$, so $f(x_2) - f(x_1) = 0$, i.e., $f(x_1) = f(x_2)$. Therefore, f is constant. $\square$

Suppose that g and h are continuous on $a \leq x \leq b$ and differentiable on $a < x < b$, and that $g'(x) \leq h'(x)$ for $a < x < b$.

- If $g(a) = h(a)$, then $g(x) \leq h(x)$ for $a \leq x \leq b$.
- If $g(b) = h(b)$, then $g(x) \geq h(x)$ for $a \leq x \leq b$.

Problem.

Explain why, if $f'(x) \leq 1$ for all x and $f(0) = 0$, that $f(x) \leq x$ for all $x \geq 0$.

Calculation Exam Prep

CALCULATION EXAM 2: FALL 2022

1. $f(x) = 8x^4 + ex^3 + 2x^{\frac{1}{2}} + 1 \Rightarrow f'(x) = 32x^3 + 3ex^2 + \frac{1}{2} \cdot 2x^{-1/2}$

2. $g(y) = 12^5 - \ln(y) \Rightarrow g'(y) = -\frac{1}{y}$

3. $h(t) = \dfrac{\csc(t) + \csc(t)t^2}{1 + t^2} = \csc(t) \Rightarrow h'(t) = -\csc(t)\cot(t)$

4. $l(s) = \sin(s)\cos(s) \Rightarrow l'(s) = \cos(s)\cos(s) - \sin(s)\sin(s)$

5. $k(w) = \frac{2w^3+3w}{3w^2+2w} \Rightarrow k'(w) = \frac{(3w^2+2w)(6w^2+3)-(2w^3+3w)(6w+2)}{(3w^2+2w)^2}$

CALCULATION EXAM 2: FALL 2022

6. $f(y) = \arctan(\sin(y)) \Rightarrow f'(y) = \frac{1}{1+(\sin(y))^2}\cos(y)$

7. $g(x) = e^{\ln(x)x} \Rightarrow g'(x) = e^{\ln(x)x}\left(\frac{1}{x}x + \ln(x)\right)$

8. $h(t) = \sqrt[4]{t^2 - 12t + \pi} + (\sin^2(t) + 21)^{-7} \Rightarrow h'(t) =$
$\frac{1}{4}\left(t^2 - 12t + \pi\right)^{-3/4}(2t - 12) + (-7)(\sin^2(t) + 21)^{-8}(2\sin(t)\cos(t))$

9. $l(s) = 12^{e^{\tan(s)}} \Rightarrow l'(s) = \ln(12)12^{e^{\tan(s)}}e^{\tan(s)}\sec^2(s)$

10. $k(w) = \frac{(4-w)^2(1+2w)}{(3+9w)} \Rightarrow k'(w) = \frac{(3+9w)[(4-w)^2(2)+(1+2w)\cdot 2(4-w)(-1)]-(4-w)^2(1+2w)(9)}{(3+9w)^2}$