§3.3: USING DERIVATIVES TO IDENTIFY EXTREME VALUES

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Lecture 18

How can we use derivatives to optimize functions?

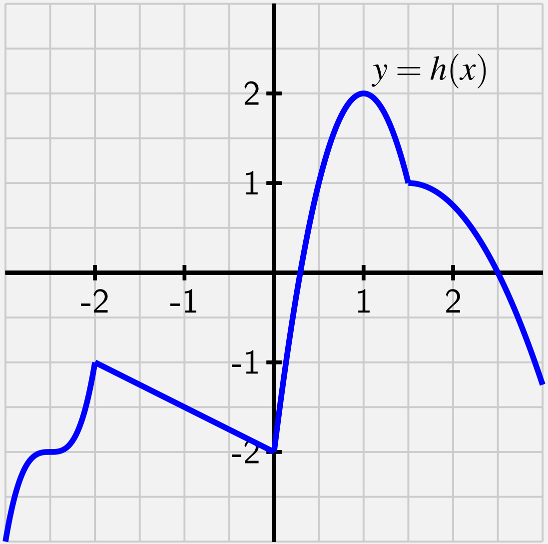
TWO TYPES OF EXTREME VALUES (DESMOS)

Definition

Given a function $y = f(x)$, we say $f(x)$ has

- *an absolute (global) maximum at $x = c$ if $f(c) \geq f(x)$ for every x in the domain of f.*
- *a relative (local) maximum at $x = c$ if $f(c) \geq f(x)$ for every x in the domain of f near c.*

Similar definitions can be given for absolute minima and relative minima. We will sometimes call these points absolute (relative) extrema.



$y = h(x)$
2
1
-1
-2
-2
-1
1
2

CRITICAL NUMBERS

Question: extrema, where are they?

Definition

Given a function $y = f(x)$, a critical number c of f is a number in the domain of f for which $f'(c) = 0$ or $f'(c)$ is undefined.

Theorem

Let p be a critical number in the domain of a function $f(x)$ and suppose $f(x)$ is differentiable near p. Then:

- *f has a relative maximum at p if and only if $f'(x)$ changes sign from positive to negative at $x = p$*
- *f has a relative minimum at p if and only if $f'(x)$ changes sign from negative to positive at $x = p$*

Example. Let's find and classify critical numbers of $f(x) = x^2e^x$.

ACTIVITY 3.3.2

Suppose that $g(x)$ is a function continuous for every value of $x \neq 2$ whose first derivative is $g'(x) = \frac{(x+4)(x-1)^2}{x-2}$. Further, assume that it is known that g has a vertical asymptote at $x = 2$.

(a) Determine all critical numbers of g. $g'(x) = 0$ implies that $x = -4$ or $x = 1$.

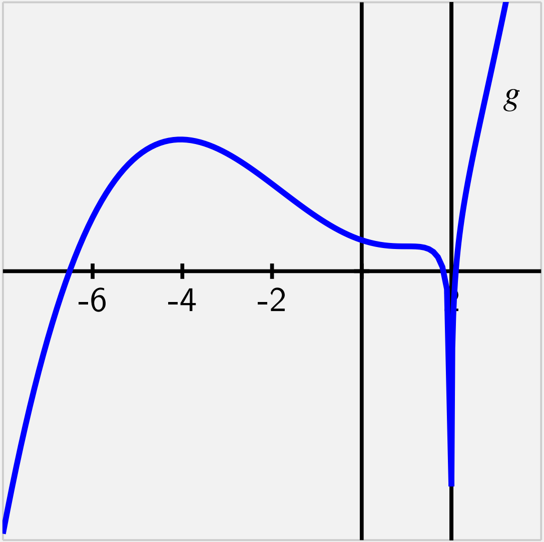
(b) By developing a carefully labeled first derivative sign chart, decide whether g has as a local maximum, local minimum, or neither at each critical number. The first derivative sign chart shows that $g'(x) > 0$ for $x < -4$, $g'(x) < 0$ for $-4 < x < 1$, $g'(x) < 0$ for $1 < x < 2$, and $g'(x) > 0$ for $x > 2$.

(c) Does g have a global maximum? global minimum? Justify your claims.

(d) What is the value of $\lim_{x\to\infty} g'(x)$? What does the value of this limit tell you about the long-term behavior of g?

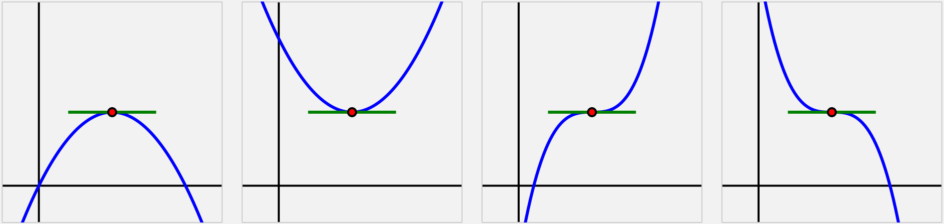
(e) Sketch a possible graph of $y = g(x)$.

ACTIVITY 3.3.2E



Day II

WHAT OF THE SECOND DERIVATIVE?



SECOND DERIVATIVE TEST

Theorem

If p is a critical number of a function f such that $f'(p) = 0$ and $f''(p) \neq 0$, then f has:

- *a relative maximum at p if and only if $f''(p) < 0$*
- *a relative minimum at p if and only if $f''(p) > 0$*

Definition

If p is a value in the domain of a continuous function $f(x)$ at which f changes concavity, we say the point $(p, f(p))$ is a point of inflection of f.

Example (Desmos). Let's do everything we can with the function

$$f(x) = \frac{1}{3}x^3 + x^2 - 3x + 4.$$

ACTIVITY 3.3.3

Suppose that g is a function whose second derivative, g'', is given by the following graph.

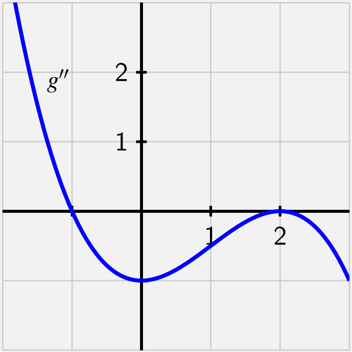


Figure 1: The graph of $y = g''(x)$.

(a) Find all points of inflection of g. $x = -1$

(b) Fully describe the concavity of g by making an appropriate sign chart.

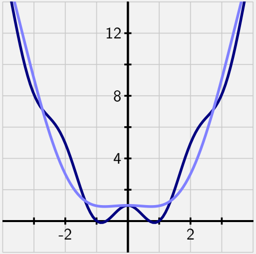
(c) Suppose you are given that $g'(-1.67857351) = 0$. Is there is a local maximum, local minimum, or neither (for the function g) at this critical point of g, or is it impossible to say? Why? Local minimum by SDT

(d) Assuming that $g''(x)$ is a polynomial (and that all important behavior of g'' is seen in the graph above, what degree polynomial do you think $g(x)$ is? Why? Probably degree 5

ACTIVITY 3.3.4

Consider the family of functions given by $h(x) = x^2 + \cos(kx)$, where k is an arbitrary positive real number.

(a) Use a graphing utility to sketch the graph of h for several different k-values, including $k = 1, 3, 5, 10$. Plot $h(x) = x^2 + \cos(3x)$ on the axes provided below.



What is the smallest value of k at which you think you can see (just by looking at the graph) at least one inflection point on the graph of h?

(b) Explain why the graph of h has no inflection points if $k \leq \sqrt{2}$, but infinitely many inflection points if $k > \sqrt{2}$.

(c) Explain why, no matter the value of k, h can only have finitely many critical numbers.