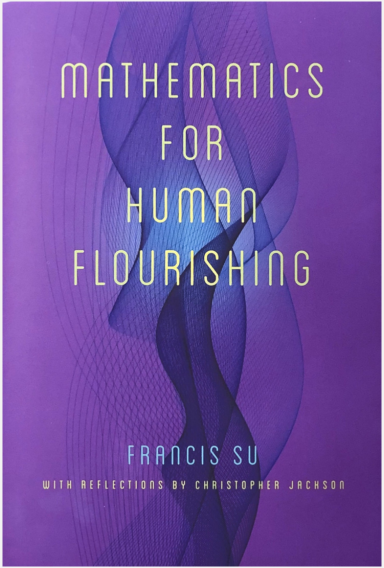
§3.4: USING DERIVATIVES TO DESCRIBE FAMILIES OF FUNCTIONS

Dr. Janssen

Lecture 19

FRANCIS SU: POWER THROUGH ABSTRACTION AND GENERALIZATION



ACTIVITY 3.4.2

Consider the family of functions defined by $p(x) = x^3 - ax$, where $a \neq 0$ is an arbitrary constant.

(a) Find $p'(x)$ and determine the critical numbers of p. How many critical numbers does p have?

(b) Construct a first derivative sign chart for p. What can you say about the overall behavior of p if the constant a is positive? Why? What if the constant a is negative? In each case, describe the relative extremes of p.

(c) Find $p''(x)$ and construct a second derivative sign chart for p. What does this tell you about the concavity of p? What role does a play in determining the concavity of p?

(d) Without using a graphing utility, sketch and label typical graphs of $p(x)$ for the cases where $a > 0$ and $a < 0$. Label all inflection points and local extrema.

(e) Finally, use a graphing utility to test your observations above by entering and plotting the function $p(x) = x^3 - ax$ for at least four different values of a. Write several sentences to describe your overall conclusions about how the behavior of p depends on a.

LEARNING TARGET

Given a family of functions, answer questions about the function and its derivative.

Let $f(x) = ax^4 + \frac{4}{3}x^3$, where $a \neq 0$.

(a) Find the critical numbers of f; your answer should include a formula in terms of a.

(b) Compute f'' and find all *possible* points of inflection.

(c) By making additional assumptions about a, use the second derivative test to classify the critical numbers (where possible).

(d) Determine whether each possible point of inflection is actually a point of inflection.

ACTIVITY 3.4.3

Consider the two-parameter family of functions of the form $h(x) = a(1 - e^{-bx})$, where a and b are positive real numbers.

(a) Find the first derivative and the critical numbers of h. Use these to construct a first derivative sign chart and determine for which values of x the function h is increasing and decreasing.

(b) Find the second derivative and build a second derivative sign chart. For which values of x is a function in this family concave up? concave down?

(c) What is the value of $\lim_{x\to\infty} a(1 - e^{-bx})$? $\lim_{x\to-\infty} a(1 - e^{-bx})$?

(d) How does changing the value of b affect the shape of the curve?

(e) Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function h and how this behavior depends on a and b.

ACTIVITY 3.4.4

Let $L(t) = \frac{A}{1+ce^{-kt}}$, where A, c, and k are all positive real numbers.

(a) Observe that we can equivalently write $L(t) = A(1 + ce^{-kt})^{-1}$. Find $L'(t)$ and explain why L has no critical numbers. Is L always increasing or always decreasing? Why?

(b) Given the fact that

$$L''(t) = Ack^2e^{-kt}\frac{ce^{-kt} - 1}{(1 + ce^{-kt})^3},$$

find all values of t such that $L''(t) = 0$ and hence construct a second derivative sign chart. For which values of t is a function in this family concave up? concave down?

(c) What is the value of $\lim_{t\to\infty} \frac{A}{1+ce^{-kt}}$? $\lim_{t\to-\infty} \frac{A}{1+ce^{-kt}}$?

(d) Find the value of $L(x)$ at the inflection point found in (b).

(e) Without using a graphing utility, sketch the graph of a typical member of this family. Write several sentences to describe the overall behavior of a typical function L and how this behavior depends on A, c, and $k$critical number.

(f) Explain why it is reasonable to think that the function $L(t)$ models the growth of a population over time in a setting where the largest possible population the surrounding environment can support is A.