MATH 390: DEVELOPMENTS IN ALGEBRA

FERMAT, DESCARTES, AND ANALYTIC GEOMETRY

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Lecture 15

PLAN/CONTEXT

- Roughly late 1500s-1650s
- Printing well-established, so the pace of dissemination of mathematical innovation had accelerated
- Oughtred, Harriot, Girard, and others turned Viète's notation into something we would recognize
- Viète's analysis is applied to geometry, creating ANALYTIC GEOMETRY, primarily developed by Pierre de Fermat and René Descartes

Advances in Notation

THE BEGINNING

- Cultural mandate: fill the earth and subdue it, have dominion over it.
- Genesis 2:19-20: God brings the animals to Adam to see what he would name them.
- Naming things is a part of who we are created to be!

WHITEHEAD

> *By relieving the brain of all unnecessary work,* ***a good notation sets it free to concentrate on more advanced problems****, and in effect increases the mental power of the [person]. Before the introduction of the Arabic notation, multiplication was difficult, and the division even of integers called into play the highest mathematical faculties. Probably nothing in the modern world would have more astonished a Greek mathematician than to learn that, under the influence of compulsory education, a large proportion of the population of Western Europe could perform the operation of division for the largest numbers. This fact would have seemed to him a* ***sheer impossibility****. The consequential extension of the notation to decimal fractions was not accomplished till the seventeenth century. Our modern power of easy reckoning with decimal fractions is the almost miraculous result of the gradual discovery of a perfect notation.*

–Alfred North Whitehead, *An Introduction to Mathematics* (1911)

THE NEED FOR AN ADVANCE

- Previously chronicled the solution to the cubic
- Cardano impeded by lack of convenient notation
- Viète impeded by restricting himself to positive solutions

SETTING THE MIND FREE: OUGHTRED AND HARRIOT

- William Oughtred (1575-1660): mathematical problems should be translated into symbolic equations and then solved via algebra
- Oughtred introduced many symbols in *Clavis Mathematicae*, particularly $\times$ for multiplication
- Thomas Harriot (1560-1621): Used $=, <, >$ as we would, as well as our symbols for square/cube roots

ALBERT GIRARD (1595-1632)

- Born in France, spent much of his life in the Netherlands
- In *A New Discovery in Algebra*:
  - Fractional exponents for roots of powers
  - Higher root notation, e.g,. $\sqrt[3]{}$
  - Fundamental Theorem of Algebra

Analytic Geometry: Fermat and Descartes

CONTEXT

- Analytic geometry: applying algebra to geometry
- ‘Born’ in 1637 after a long gestation period
- Note the need for everything that came before, particularly notation required for working creatively with complex ideas
- Roughly simultaneous discovery by Fermat and Descartes, but each approach to analytic geometry bears its creator’s image

PIERRE DE FERMAT (1601-1665)

- Studied law at Toulouse, Bordeaux, and Orléans, where he also studied mathematics with students of Viète in the 1620s
- Seems to have been an average lawyer, perhaps because he spent so much time working on mathematics
- Best known today for his Last Theorem, written in the margins of Diophantus' *Arithmetica*

INTRODUCTION TO PLANE AND SOLID LOCI

- Early 1637, Fermat shared a manuscript, *Ad locos planos et solidos isagoge*
- Attempt to reconstruct the *Plane Loci* of Apollonius with an algebra-first perspective
- Contains the germ of the two major ideas of analytic geometry:
    1. The correspondence between geometric loci and indeterminate algebraic equations in two or more variables
    2. The geometric framework for this correspondence: a system of axes along which lengths are measured

ONE AXIS

- Begin with a fixed straight line (an axis)
- What curves are traced out by the motion of a line segment of variable length, one end of which is on our fixed line?

THEOREM

Let NZM be a straight line given in position, with point N fixed. Let NZ be equal to the unknown quantity A, and ZI, the line drawn to form the angle NZI, the other unknown quantity E. If D times A equals B times E, the point I will describe a straight line given in position.

Note that Fermat, as with most others, restricted himself to positive numbers.

Theorem

If Aq equals D times E, point I lies on a parabola.

He knew it wasn't the full parabola.

Was then able to determine the curves represented by five other quadratics in two variables assuming the constructions of the conic sections in Apollonius.

RENÉ DESCARTES (1596-1650)

- Philosopher, mathematician
- *Discourse on Method* (1637)
- Seen as responsible for increased interest in epistemology in 17th century
- Cartesian coordinates named for him
- *Geometry*: appendix written to demonstrate methods of reasoning discussed in the *Discourse*

GEOMETRY

- "Any problem in geometry can easily be reduced to such terms that a knowledge of the lengths of certain straight lines is sufficient for its construction."
- "[O]ften it is not necessary thus to draw the lines on paper but it is sufficient to designate each by a single letter."
- Represented squares as line segments!
- EXAMPLE: solving $z^2 = az + b^2$

PROBLEM OF THE FOUR LINES

- From Apollonius: Given three fixed straight lines, find the locus of a point moving so that the square of its distance to one line is in a constant ratio to the product of its distance to the other two lines.
- Descartes introduces a coordinate axis to solve this problem.
- The equation describing the curve of such points is quadratic; Descartes shows that it's either a circle or one of the other conic sections
- Then Descartes goes further and uses his method to solve a five-line problem

- States a version of Girard's result (FTA)
- Shows how equations are built from their solutions
- Descartes Rule of Signs
- Factor Theorem

COMPARING FERMAT AND DESCARTES

- Descartes: Geometry $\rightarrow$ algebra
- Fermat: Algebra $\rightarrow$ geometry
- This distinction required Descartes to deal with more complicated polynomials
- However, it forced him to develop methods of dealing with polynomials of high degree
- Fermat never published his work
- Descartes did, but it was difficult to read; gaps intentionally left to "leave to others the pleasure of discovery"