MATH 390: PROBABILITY

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Lecture 20

EARLY WORK: PASCAL AND HUYGENS

Games of chance: what is "fair" in an interrupted game?

THEOREM (PASCAL)

Suppose that the first player lacks r games of winning the set while the second player lacks s games, where both r and s are at least 1. If the set of games is interrupted at this point, the stakes should be divided so that the first player gets that proportion of the total as $\sum_{k=0}^{s-1} \binom{n}{k}$ is to 2^n, where $n = r + s - 1$ (the maximum number of games left).

Christian Huygens (1629-1695) extended this to EXPECTATION: "To have equal chances of winning a or b is worth $(a+b)/2$ to me.

JAKOB BERNOULLI

- Impossible to enumerate all possible sources of risk
- Instead, look backward at results from similar instances
- That is, consider some statistics
- It seemed to him that the more observations one made, the better one's predictions
- Proved the Law of Large Numbers, which appeared posthumously in the *Ars Conjectandi* (1713)

THE ARS CONJECTANDI

- Part I: commentary on Huygens' 1657 *De Ratiociniis in aleae ludo*, with more generality or better solutions
- Part II: Developed laws of permutations and combinations
- Part III: Applied his results to games of chance
- Part IV: *The Use and Application of the Preceding Doctrine in Civil, Moral, and Economic Matters*.
  - Shows how to determine probabilities of "real-life" situations *a posteriori*
  - Introduced the ideas of moral certainty and moral impossibility

BERNOULLI'S LAW OF LARGE NUMBERS

- Suppose we have a random process for which N observations are made, X of which are successes
- We wish to determine the true probability p of success
- Given any small number $\epsilon > 0$ and any large number $c \gg 0$, a number $N = N(c)$ may be found so that the probability that X/N differs from p by no more than ϵ is greater than c times the probability that X/N differs from p by no less than ϵ:

$$P\left(\left|\frac{X}{N} - p\right| \leq \epsilon\right) > cP\left(\left|\frac{X}{N} - p\right| > \epsilon\right)$$

BERNOULLI'S CALCULATION

Considered the theorem "virtually intuitive" so wanted to contribute by providing methods of calculating $N(c)$.

If the true probability is $p = r/t$, with $t = r + s$, and we desire moral certainty, $c = 1000$, Bernoulli showed that $N(c)$ could be taken to be any integer greater than the larger of

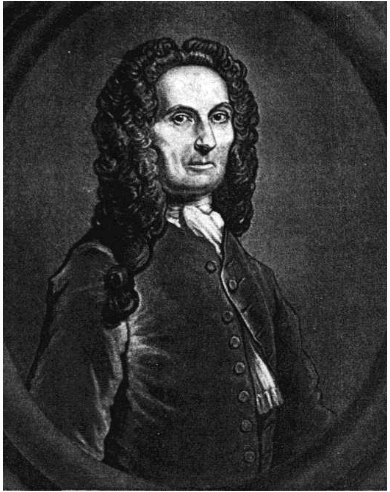
$$mt + \frac{st(m-1)}{r+1} \qquad \text{and} \qquad nt + \frac{rt(n-1)}{s+1},$$

where m, n are integers satisfying

$$m \geq \frac{\log c(s-1)}{\log(r+1) - \log r} \qquad \text{and} \qquad n \geq \frac{\log c(r-1)}{\log(s+1) - \log s}.$$

ABRAHAM DE MOIVRE (1667-1754)

- French mathematician and probabilist
- Wrote *The Doctrine of Chances* (1718, 1738, 1756)
- First to give a precise definition of probability
- Noted the sum of complementary events
- Used infinite series to perform his calculations, especially those involving logarithms



REV. THOMAS BAYES (1702-1761)

- *An Introduction to the Doctrine of Fluxions, and a Defence of the Mathematicians Against the Objections of the Author of The Analyst* (1736)
- *An Essay towards Solving a Problem in the Doctrine of Chances* (1764)
- Problem:

  > *Given the number of times in which an unknown event has happened and failed. Required the chance that the probability of its happening in a single trial lies somewhere between any two degrees of probability that can be named.*

- That is, if X represents the number of times the event has happened in n trials, x the probability of its happening in a single trial, and r and s the two given probabilities, Bayes wants

$$P(r < x < s|X)$$

BAYES' PROPOSITIONS

Let E and F be subsequent events.

Proposition (Proposition 3)

The probability of both events $P(E \cap F)$ is

$$P(E \cap F) = P(E)P(F|E).$$

Proposition (Proposition 5 (Bayes' Theorem))

The probability of E given F is

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

APPLICATION: ANNUITIES

- Annuities sold for centuries
- Buyer (annuitant) considered it a bet, while the seller considered it a loan at interest
- Before De Moivre, terms were set based on experience of the parties or need of cash for the seller.
- English law of 1540: a government annuity is worth seven years' purchase.