MATH 390: MODERN ALGEBRA

THE INSOLUBILITY OF THE QUINTIC AND THE STRUCTURAL REVOLUTION

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Lecture 22

The Insolubility of the Quintic

CONTEXT

- Italian algebraists found a general form for the solutions of both cubics and quartics
- Suspicion grew over the years that the quintic did not have such a solution
- Paolo Ruffini (1765-1822) presented the first proposed proof, but apparently no one could understand it

NIELS HENRIK ABEL (1802-1829)

- Norwegian
- First believed quintic was solvable by radicals
- Eventually proved it wasn't
- Died young of TB, two days before a position was secured for him in Berlin
- Commutative groups called ABELIAN in his honor

EVARISTE GALOIS (1811-1832)

- French
- Failed entrance exams to the École Polytechnique and enrolled in the École Normale
- Arrested twice and jailed for six months, during which he completed his memoir on the solvability of equations
- Killed in a duel 5 months before turning 21

GALOIS' IDEAS

Proposition

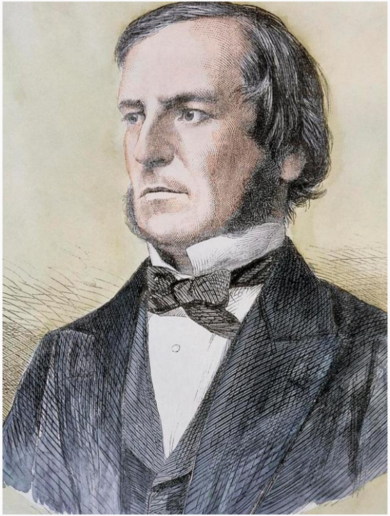
Let an equation be given of which $a, b, c, \ldots$ are the m roots. There will always be a group of permutations of the letters $a, b, c, \ldots$ which has the following property:

1. *that every function of the roots, invariant under the substitutions of the group, is rationally known*
2. *conversely, that every function of the roots which is rationally known is invariant under the substitutions*

Galois' results were eventually published in 1846 by Liouville.

GEORGE BOOLE (1815-1864)

- English mathematician, not much formal schooling beyond primary school
- Became a professor at Cork in Ireland
- Algebracization of logic laid the groundwork for circuit design



BOOLE'S PROPOSITION 1

Proposition

All the operations of Language, as an instrument of reasoning, may be conducted by a system of signs composed of the following elements, viz.:

1. *Literal symbols, as x, y, etc., representing things as subjects of our conceptions*
2. *Signs of operation, as $+$, $-$, $\times$, standing for those operations of the mind by which the conceptions of things are combined or resolved so as to form new conceptions involving the same elements*
3. *The sign of identity, $=$.*

And these symbols of Logic are in their use subject to definite laws, partly agreeing with and partly differing from the laws of the corresponding symbols in the science of Algebra.

Structure: Groups, Fields, and Rings

ABSTRACT DEFINITION OF A GROUP

- Arthur Cayley, *On the Theory of Groups* (1854)
- Abstracted permutations to any collection of functions; developed further in four papers in 1878
- Leopold Kronecker defined a similar system in 1870, seemingly unaware of Cayley's work
- Dedekind also abstracted from Galois' work in lectures in 1856
- In 1882, Walther von Dyck and Heinrich Weber developed the major ideas of what is now group theory

THE CONCEPT OF A FIELD

- Implicit in Galois' work in 1830
- Kronecker built fields out of the whole numbers (1850)
- Irrational quantities like $\sqrt{2}$ made no sense without a way to construct them out of the whole numbers
- Weber combined the Dedekind-Kronecker approach with the finite systems of Galois to define a field in general in 1893

STRUCTURAL REVOLUTION

- Weber gave these first modern versions of important structural definitions
- What is structure?
- Yet he still understood the connection to solutions of equations, and presented them to his students this way

- Competing notions developed by Hilbert, Hensel, and Fraenkel
- Modern definition given by Emmy Noether: a set R with two operations, $+$ and $\times$, such that R is a commutative group under addition, multiplication is associative, and the distributive laws hold.
- An important class of rings is named in her honor

EMMY NOETHER (1882–1935)

- German-born
- Second woman to receive a doctorate from the University of Erlangen
- Moved to Göttingen to work with Hilbert and Klein; was not paid for her lectures
- Made substantial contributions to commutative algebra and physics