MATH 390: AXIOMS AND SET THEORY

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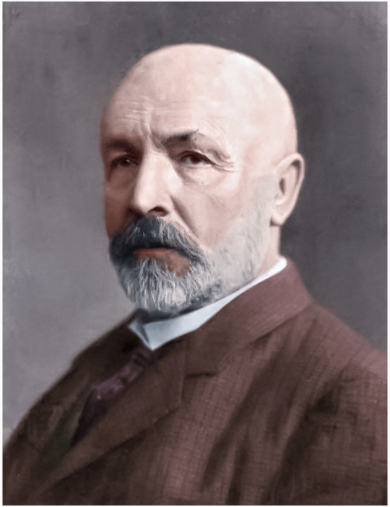
Lecture 24

Set Theory

GEORG CANTOR (1845-1918)

- Son of a Lutheran pastor
- Known for work on set theory and infinite cardinals
- Faced opposition for his theory of infinite sets
- Organized the Association of German Mathematicians in 1890, and the first ICM in 1897



INTRO TO THE INFINITE

- A number of mathematicians became interested in rigorously defining the real numbers to finally place calculus on a firm mathematical foundation
- One approach due to Dedekind (cuts!)
- Others due to Meray, Kossak, Heine, and Cantor, who defined real numbers in terms of **Cauchy sequences** of rational numbers
- Considering the set of limit (accumulation) points of an infinite set led to Cantor developing a theory of **transfinite** ordinal numbers

QUESTION TO DEDEKIND (1873)

Take the collection of all positive whole numbers n and denote it by (n); then think of the collection of all real numbers x and denote it by (x); the question is simply whether (n) and (x) may be corresponded so that each individual of one collection corresponds to one and only one of the other? … As much as I am inclined to the opinion that (n) and (x) permit no such unique correspondence, I cannot find the reason.

A NEW FRAMEWORK FOR SET THEORY

Every set M has a definite 'power,' which we will also call its 'cardinal number.' We will call by the name 'power' or 'cardinal number' of M the general concept which, by means of our active faculty of thought, arises from the set M when we make abstraction of the nature of its various elements m and of the order in which they are given.

- $\mathbb{N}$ has cardinality $\aleph_0$
- $\mathbb{R}$ has cardinality $\mathcal{C}$

REACTION

- Leopold Kronecker: any mathematical construction must be capable of completing in a finite number of operations
- As editor of a prestigious journal, Kronecker was able to hold up publication of some of Cantor's work
- Conflict continues to this day

Building Numbers

PEANO'S AXIOMS FOR $\mathbb{N}$ (1889)

1. 0 is a natural number
2. For every natural number n, $S(n)$ is a natural number
3. For all natural numbers m and n, $m = n$ if and only if $S(m) = S(n)$
4. For every natural number n, $S(n) = 0$ is false
5. If K is a set such that $0 \in K$ and for every natural number n, $n \in K$ implies $S(n) \in K$, then K contains every natural number

A MODEL FOR $\mathbb{N}$

Let S be an operator on sets defined as $S(A) = A \cup \{A\}$. Then:

- $0 = \varnothing$
- $1 = S(0) = S(\varnothing) = \varnothing \cup \{\varnothing\} = \{0\}$
- $2 = S(1) = S(\{0\}) = \{0\} \cup \{\{0\}\} = \{0, \{0\}\} = \{0, 1\}$
- Etc.

Then this set-theoretic model, together with the successor function S defined above satisfies Peano's axioms

Consider the set

$$Z = \{(a, b) \, : \, a, b \in \mathbb{N}\}$$

under the **equivalence relation** $(a, b) \sim (c, d)$ if and only if $a + d = b + c$.

Consider the set

$$Q = \{(a, b) : a, b \in \mathbb{Z}, b \neq 0\}$$

under the relation $(a, b) \sim (c, d)$ if and only if $ad = bc$.

Is it the case that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}?$$

Axioms of Set Theory

WELL ORDERING

- Cantor realized that the **trichotomy principle** was equivalent to the idea that every set can be **well ordered**
- 1883: Believed that $\mathbb{R}$ was well ordered (though obviously not under the usual order), but by the mid-1890s he realized this needed proof, and eventually realized his 1897 proof was incomplete
- Presented by David Hilbert as a problem at the 1900 ICM in Paris

PARADOXES IN SET THEORY

- Cantor believed that virtually any description of "objects of our thought" would define a set
- Cantor did not try to axiomatize set theory, which led to paradoxes
- Russell's Paradox (1903) placed limits on the kinds of sets that could be defined without contradiction

ZERMELO AND THE AXIOM OF CHOICE

1904: Ernst Zermelo proves the well-ordering theorem based on the **axiom of choice**.

Let M be a set and S the collection of all nonempty subsets of M. Then there exists a function $\gamma : S \to M$ such that $\gamma(X) \in X$.

That is, it is possible to simultaneously choose an element of every nonempty subset of M, even if M is infinite.

Seems obvious, and had been implicitly used for decades, but raised controversy, again around performing an infinite number of actions all at once. Controversy was enhanced as this enabled the proof of a result of which many were skeptical. What was needed were axioms for set theory.

CONSEQUENCES

Some useful, some weird

- Every vector space has a basis
- Zorn’s Lemma
- Banach-Tarski