Mathematics 1900-Today

Lecture 28

Major themes

- Professionalization
- Collaboration
- Major Results Proved
- Open Questions

Professionalization

Working mathematicians

- For the most part, mathematicians work in u
  in BIG fields
- National Security Agency: the largest single e
  the world (?)
- Mathematicians founded societies to advance

American Mathematical Society (AMS)

- Mission Statement: The AMS, founded in 1888 to further the interests of mathematical research and scholarship, serves the national and international community through its publications, meetings, advocacy and other programs, which
  - promote mathematical research, its communication and uses,
  - encourage and promote the transmission of mathematical understanding and skills,
  - support mathematical education at all levels,
  - advance the status of the profession of mathematics, encouraging and facilitating full participation of all individuals,
  - foster an awareness and appreciation of mathematics and its connections to other disciplines and everyday life.

Mathematical Association of America (1915)

- The mission of the MAA is to advance the understanding of mathematics and its impact on our world.
- MAA's core values are:
  - Community
  - Inclusivity
  - Communication
  - Teaching & Learning

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

- Founded 1920
- Mission: The National Council of Teachers of Mathematics advocates for high-quality mathematics teaching and learning for each and every student.
- Strategic framework:
    - Teaching and Learning
    - Access, Equity, and Empowerment
    - Building Member Value
    - Advocacy

Society for Industrial and Applied Mathematics

- Founded 1951
- Dedicated to the use of mathematics in industry
- The world's largest professional organization devoted to applied mathematics

Society for Industrial and Applied Mathematics

Association for Women in Mathematics

- Founded in 1971
- AWM Mission: The purpose of the Association for Women in Mathematics is to encourage women and girls to study and to have active careers in the mathematical sciences, and to promote equal opportunity and the equal treatment of women and girls in the mathematical sciences.

Association of Christians in the Mathematical Sciences

- Founded 1971
- The Association of Christians in the Mathematical Sciences developed initially from a desire on the part of a group of mathematics faculty at Christian colleges to integrate their faith with their academic discipline.

Mathematics Genealogy Project

- "The intent of this project is to compile information about ALL the mathematicians of the world. We earnestly solicit information from all schools who participate in the development of research level mathematics and from all individuals who may know desired information."
- Started in the '90s by Harry Coonce at MSU-Mankato
- https://www.mathgenealogy.org

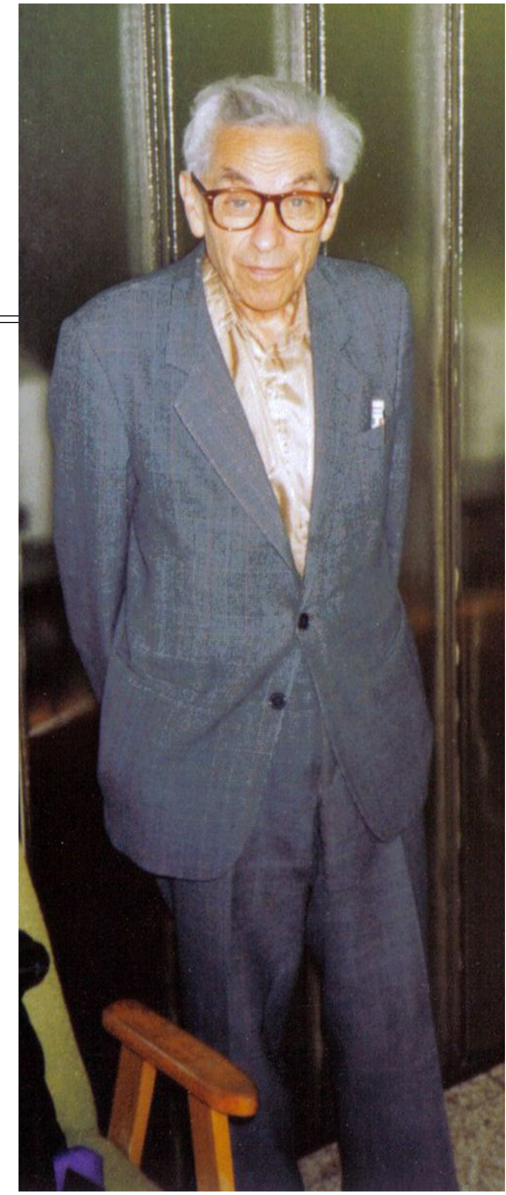
Collaboration

Nicholas Bourbaki

- Collective of young French mathematicians founded in 1934
- Working on The Elements of Mathematics, a reshaping of mathematics in extremely abstract terms. Most recent volume published in 2016
- Goal to ground all of mathematics in set theory

Paul Erdös

- 1913—1996, Hungarian mathematician
- Incredibly prolific: worked with over 500 collaborators and published 1500 papers
- Spent his life living out of a suitcase and traveling to visit collaborators
- Erdös number: degree of separation from Erdös (who alone has an Erdös number of 0)



Polymath Project

- Started on Timothy Gowers' blog in 2009 — posed a problem and asked readers to post partial answers/progress toward a solution
- "Is massively collaborative mathematics possible?"
- Polymath8a and Polymath8b: bounded gaps on primes

Major Results

Hilbert's Problems

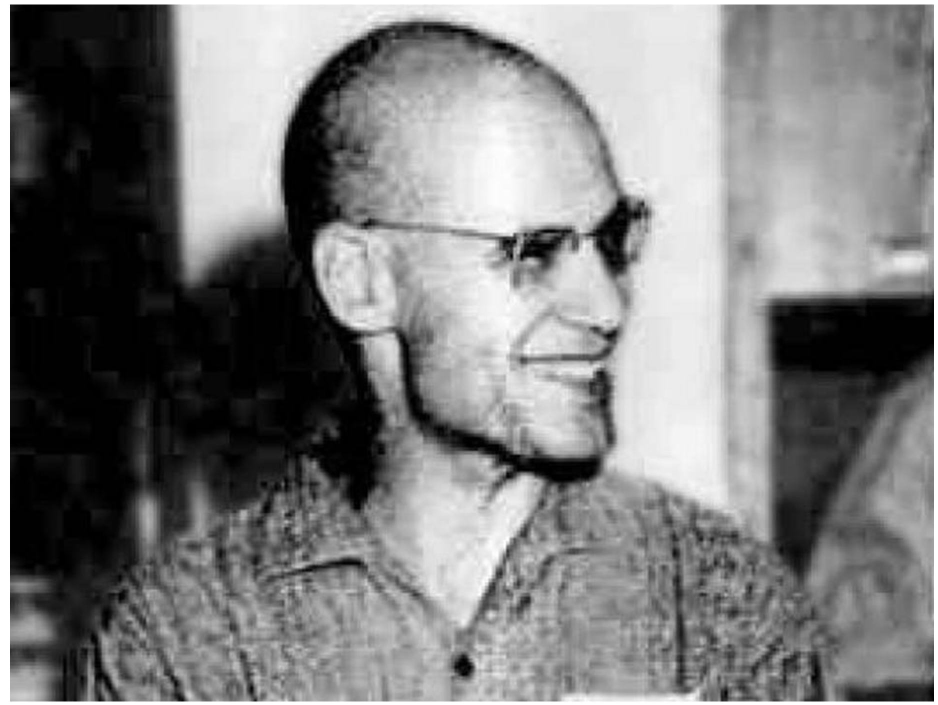
- Delivered at the 1900 ICM
- 23 problems to guide mathematical work in the twentieth century
- Problems 1—2 concern axioms (CH and consistency)

Emmy Noether

- 1882—1935
- With Wolfgang Krull, gave the first axiomatic description of a commutative ring
- Also proved Noether's Theorem in physics which established a one-to-one correspondence between symmetries and conservation laws

Alexander Grothendieck

- 1928—2014
- Fields medalist (1966), but refused to travel to Moscow to receive it
- Revolutionized algebraic geometry through abstraction; many famous conjectures became corollaries to his theorems



Category Theory

- For nearly all mathematical objects (sets, rings, groups, vector spaces, graphs, etc), there are natural functions to define
- Category theory (introduced by Mac Lane and Eilenberg in 1945) formalizes these similarities
- Study of *arrows* (or *morphisms*) between *objects*
- Functors take categories to categories
- Natural transformations take functors to functors (while respecting internal structure)

Topology

- Study of the properties of geometric objects preserved under continuous deformation (stretching, twisting, crumpling, bending, but not tearing or gluing)

- "Rubber sheet geometry"

- Algebraic topology: attaching natural algebraic structures to topological spaces; introduced by Poincaré in 1895

- Applications to biology, computer science, physics, data analysis, others

Four Color Theorem

- The problem
- That five colors sufficed was known in the 1800s
- Alfred Kempe gave a “proof” in 1879; flaws discovered in 1890
- 1960s-1970s, Heinrich Heesch developed methods of using computers to attack the problem
- June 21, 1976: Kenneth Appel and Wolfgang Haken, UIUC
- Basic idea: if the four-color conjecture were false, there would be at least one map with the smallest number of regions that required five colors (WOP)
- They reduced to 1,834 cases, which had to be checked one at a time by a computer, and took over 1000 hours.

Bounded Gaps on Primes

- Twin Prime Conjecture: there are infinitely many pairs of primes $p > q$ such that $p - q = 2$

- Conjecture is so old we have no idea who made it; could go back to the Greeks

- For centuries, no one even knew if there was a number b for which there were infinitely many pairs of primes $p > q$ such that $p - q \leq b$

Yitang Zhang

- 1955—
- First finite bound on the least gap between consecutive primes attained infinitely often
- Zhang's techniques, combined with some discovered by the PolyMath8 project, pushed the gap down to 246
- Best attainable bound with current methods is 6

Open questions

Millennium Problems (May 2000)

- Yang-Mills Mass Gap
- Riemann Hypothesis
- P vs NP
- Navier-Stokes
- Hodge Conjecture
- ~~Poincaré Conjecture~~ (Solved by Grigori Perelman in 2003)
- Birch and Swinnerton-Dyer Conjecture

Prime Number Theorem (1896)

- Let $\pi(x)$ be the number of primes less than or equal to a real number x.
- Example: $\pi(20) = 8$
- **PNT:** $\displaystyle\lim_{x\to\infty} \frac{\pi(x)}{\left[\frac{x}{\log(x)}\right]} = 1.$
- But PNT doesn't tell us about the exact difference between the two as $x \to \infty$.

Riemann Hypothesis

- Also part of Hilbert's Eighth Problem
- Riemann zeta function: $\zeta : \mathbb{C} \to \mathbb{C}$, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$
- **RH**: The real part of every non-trivial zero of $\zeta(s)$ is $\frac{1}{2}$.
- Fact: the error (difference) from PNT is related to the position of the nontrivial zeros of the zeta function.
- Many claimed solutions, most recently by Sir Michael Atiyah (1929-2019)

Other prime number problems

- Are there infinitely many ______ primes?
- Cousin, Mersenne, Germain, Pierpont, regular, Wilson, Fibonacci, etc

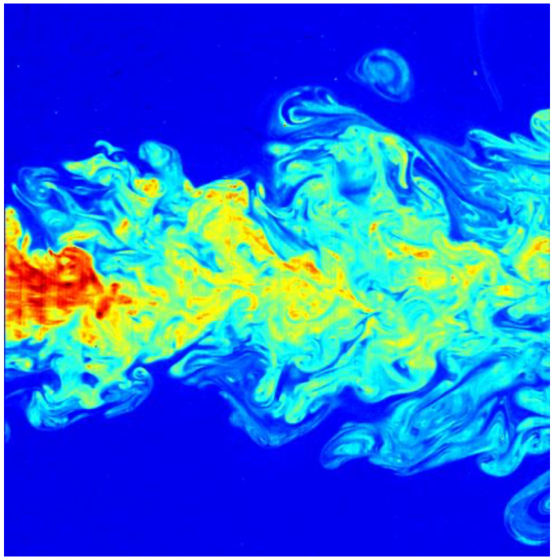
P vs NP

- A major unsolved problem in theoretical computer science
- Informally: can every problem whose solution can be verified “quickly” (**NP**) also be solved “quickly” (**P**)
- Big example: the *integer factorization* problem is in **NP** but not known to be in **P**.
- If an efficient means for solving IF is found, there will be major implications for our cryptographic systems

Navier-Stokes Equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{v} + \mathbf{f}(\mathbf{x}, t)$$

- Nonlinear partial differential equations describing the flow of a viscous fluid
- Solutions in certain situations exist, but general smooth solutions do not



The *abc* conjecture

- Proposed by Joseph Oesterlé (1988) and David Masser (1985)
- **Notation:** Given a positive integer n, $\mathrm{rad}(n) = \prod_{p|n,\ p \text{ prime}} p$.
- So, $\mathrm{rad}(13) = 13$, $\mathrm{rad}(25) = 5$, etc.
- For all $\epsilon > 0$, there are only finitely many triples (a, b, c) of relatively prime positive integers with $a + b = c$ such that $c > \mathrm{rad}(abc)^{1+\epsilon}$.

…or is it a theorem?

- August 2012: Shinichi Mochizuki claims a proof
- No one can understand it!
- March 2018: visited by Peter Scholze and Jakob Stix
- Still unresolved